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CONVERGENCE OF PROGRESSIVELY CENSORED LIKELIHOOD
RATIO PROCESSES IN LIFE-TESTING

By

Joseph C. Gardiner*
Michigan State University

Abstract

The weak convergence of certain (randomly stopped) likelihood ratio processes based on the ordered observations corresponding to a random sample is considered in the situation where the hazard rate function of the underlying distribution is separable in its variables. It is shown that under mild conditions on the stopping variables the log-likelihood function is locally asymptotically normal. Some remarks pertaining to the general ease and applications of the theorems proved are also discussed.

AMS Subject Classifications: 60B10, 60G40

Key Words and Phrases: Clinical trials, life-testing, likelihood ratio statistics, progressive censoring, stopping times, weak convergence, Wiener process.

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1. INTRODUCTION. In a variety of statistical experiments, notably those connected with clinical trials and life tests, the observable variables are time-ordered and consequently recorded in the order of increasing size. Specifically if $X_1,\ldots,X_n$ $(n > 1)$ denote the survival times of n specimens under a life test then in the typical situation encountered in practice the observable random variables correspond to the ordered sample $X_{n,1},\ldots,X_{n,n}$ based on $X_1,\ldots,X_n$ rather than on the survival times themselves. Now a complete collection of the data calls for the monitoring of the experiment until the last observation $X_{n,n}$ is recorded. However, ethical considerations and limitations on time and cost often demand curtailment of experimentation before all the specimens under study have responded and so, in practice, experimentation may be terminated after a prespecified proportion of units have responded (censoring) or alternatively after the investigation has been monitored for a prespecified length of time (truncation). These sampling procedures themselves lack certain elegancies stemming from cost and efficiency considerations. For example, a too early truncation typically increases the risk of erroneous decision whereas in the censoring scheme the randomness of the time of termination of experimentation could be at variance with restrictions on time and cost. For these reasons it is generally desirable to monitor the experiment from the onset and continuously update the data so that at any stage of the experiment if the current evidence warrants a clear statistical decision the experiment can be terminated at that stage with the adoption of the appropriate decision which the current accumulated evidence indicates. Such sampling schemes are called Progressively Censored Schemes (PCS) and in our formulation they lead naturally to the consideration of a broad class of stopping variables $\{\tau_n;\ n \geq 1\}$, where for each $n \geq 1$, τ_n is defined in terms of the observables $X_{n,1},\ldots,X_{n,n}$.

The purpose of this note is to develop an invariance principle for progressively censored likelihood ratio processes valid for a class of survival distributions in which the hazard rate function is separable in its variables. Whereas in Sen (1976) and Gardiner (1978) weak convergence results have been derived for the general case, in the present situation an immensely simplified analysis can be presented under fewer regularity conditions using simple classical techniques.

Along with the preliminary notions the main results are formulated in Section 2. Section 3 deals with the proofs of the theorems and Section 4 is devoted to some general remarks and extensions.

2. Preliminary notions and the main theorems. Let $\{X_i; i \geq 1\}$ be a sequence of independent and identically distributed random variables (iidrv) whose distribution ν_θ on the Borel line $(R,\mathcal{B})$ depends on a parameter θ, $\theta \in \Theta \subseteq R$. Assume Θ to be an open interval of R. We suppose the family of measures $\{\nu_\theta : \theta \in \Theta\}$ is dominated by Lebesgue measure μ on $(R,\mathcal{B})$ and write $f_\theta(\cdot) = d\nu_\theta/d\mu$ for a version of the probability density function (pdf) and $F_\theta(\cdot)$ for the corresponding distribution function (df), which is then continuous. Let $(R_j,\mathcal{B}_j)$, $j \geq 1$ be copies of the Borel line and set $(X,A) = \Pi_{j=1}^{\infty}(R_j,\mathcal{B}_j)$ with P_θ denoting the product measure of the ν_θ induced on A. E_θ will denote the expectation evaluated with respect to P_θ.

Following the usual terminology we define the survival function G_θ by

$$G_\theta(x) = 1 - F_\theta(x) \tag{2.1}$$

and the hazard rate function (force of mortality, intensity rate) r_θ by

$$r_\theta(x) = f_\theta(x)/G_\theta(x). \tag{2.2}$$

We suppose $f_\theta(x) > 0$ for every $x \in R$ and $\theta \in \Theta$ so that $r_\theta(x) > 0$ whenever $0 < F_\theta(x) < 1$. In this note we shall confine attention to distributions for which $r_\theta(x)$ can be expressed as

$$r_\theta(x) = h(x)/Q(\theta) \tag{2.3}$$

where h and Q are respectively functions of x and θ only. The class of univariate densities f_θ satisfying (2.3) form a subclass of the exponential family. We require h to be μ integrable over each interval $[a,b]$, $a,b \in R$ and Q to be continuously differentiable on $\overline{\Theta}$, the closure of Θ in R.

In the typical situation encountered in life testing and clinical trials the observable random variables are the order statistics $Z_1 < Z_2 < \ldots < Z_n$ corresponding to $X_1,\ldots,X_n$. We denote by

$$\underset{\sim}{Z}^{(k)} = (Z_1,\ldots,Z_k),\ 1 \le k \le n;\ Z_0 = \underset{\sim}{Z}^{(0)} = 0 \tag{2.4}$$

and write $\mathcal{B}_{n,k}$ for the σ-field generated by $\underset{\sim}{Z}^{(k)}$, $1 \le k \le n$. $\mathcal{B}_{n,0}$ is the trivial σ-field. We also consider a class of stopping variables τ_n, $n \ge 1$ such that for each n, τ_n is adapted to the σ-fields $\{\mathcal{B}_{n,k} : 1 \le k \le n\}$.

Now for every k, $1 \le k \le n$, the joint pdf of $\underset{\sim}{Z}^{(k)}$ is

$$p_\theta(\underset{\sim}{z}^{(k)},n) = \{n!/(n-k)!\}\{\prod_{i=1}^{k} f_\theta(z_i)\}\{G_\theta(z_k)\}^{n-k} \tag{2.5}$$

defined on the domain $\{\underset{\sim}{z}^{(k)} : -\infty < z_1 <\ldots< z_k < \infty\}$. Fix θ_0 in Θ and for a sequence $\{\theta_n\}$ in Θ of the form

$$\theta_n = \theta_0 + un^{-\frac{1}{2}},\ u \in R \tag{2.6}$$

we define for each k, $1 \le k \le n$

$$\Lambda_{n,k}(u) = p_{\theta_n}(\underset{\sim}{Z}^{(k)},n)/p_{\theta_0}(\underset{\sim}{Z}^{(k)},n). \tag{2.7}$$

Note that $\Lambda_{n,n}(u)$ reduces to the classical likelihood ratio statistic $\prod_{i=1}^{n}\{f_{\theta_n}(X_i)/f_{\theta_0}(X_i)\}$ for the iid sequence $X_1,\ldots,X_n$ and parameters θ_0 and θ_n. We are here concerned with the "stopped" statistics Λ_{n,τ_n}, where $\Lambda_{n,\tau_n} = \Lambda_{n,k}$ when $\tau_n = k$. We shall develop an invariance principle for the process $u \to \Lambda_{n,\tau_n}(u)$. To formulate this precisely let us write $\Theta = (a,b)$, where $-\infty \le a < b \le +\infty$, $a_n = n^{\frac{1}{2}}(a - \theta_0)$, $b_n = n^{\frac{1}{2}}(b - \theta_0)$, and define $\Lambda_{n,\tau_n}(u)$ by

$$\begin{aligned}
(2.8) \qquad \Lambda_{n,\tau_n}(u) &= p_{\theta_n}(\underset{\sim}{Z}^{(\tau_n)},n)/p_{\theta_0}(\underset{\sim}{Z}^{(\tau_n)},n), \text{ if } u \in (a_n,b_n) \\
&= (u - a_n + 1)^2\Lambda_{n,\tau_n}(a_n), \text{ if } u \in (a_n - 1,a_n], \ a_n > -\infty \\
&= 0 \qquad\qquad\qquad\qquad, \text{ if } u \le a_n - 1, \ a_n > -\infty
\end{aligned}$$

and similarly for $u \in [b_n,b_n + 1)$ and $u \ge b_n + 1$ if $b_n < +\infty$. Then Λ_{n,τ_n} has sample paths in $C(R)$ — the space of all real-valued continuous functions on R. We endow $C(R)$ with the topology of uniform convergence on compacta. Then $C(R)$ is a complete separable metric space.

Define the sequence

$$(2.9) \qquad \xi_{n,k} = \frac{\partial}{\partial\theta}(\log p_\theta(\underset{\sim}{Z}^{(k)},n)), \ k = 1,\ldots,n. \quad \xi_{n,0} = 0$$

and write

$$(2.10) \qquad J_{n,\tau_n}(\theta) = E_\theta\{\xi^2_{n,\tau_n}\}.$$

Introduce the sequence of functions $\{k_n(\cdot;\theta);\ n \ge 1\}$ on $[0,1]$ by

$$(2.11) \qquad k_n(t;\theta) = [tE_\theta(\tau_n)],$$

where $[x]$ denotes the greatest integer $\le x$. Then the k_n may assume only the integer values $0,\ldots,n$. Finally for fixed θ we define a process $W_{n,\tau_n} = \{W_{n,\tau_n}(t;\theta) : t \in [0,1]\}$ by

(2.12) $$W_{n,\tau_n}(t;\theta) = \xi_{n,k_n}(t;\theta)/J^{\frac{1}{2}}_{n,\tau_n}(\theta).$$

This process has sample paths in $D[0,1]$ -- the space of all real valued right continuous functions on $[0,1]$ with left hand limits. Equip $D[0,1]$ with the usual Skorohod topology. Write

(2.13) $$\Lambda(u) = \exp\{uJ^{\frac{1}{2}}_{\alpha}(\theta_0)\zeta - \tfrac{1}{2}u^2 J_{\alpha}(\theta_0)\}, \quad u \in R$$

where ζ is a standard Gaussian variable and

(2.14) $$J_{\alpha}(\theta) = \alpha(Q'(\theta)/Q(\theta))^2, \quad \theta \in \Theta,\ \alpha \in (0,1].$$

Theorem 1. *With the assumptions made above and* θ *fixed in* Θ, *whenever* $n^{-1}\tau_n \to \alpha \in (0,1]$ *in* P_θ*-probability and* $Q'(\theta) \neq 0$, then under P_θ,

(2.15) $$W_{n,\tau_n} \to_w W \ \text{in}\ D[0,1],$$

where $W = \{W(t) : t \in [0,1]\}$ *is a standard Wiener process in* $D[0,1]$.

Theorem 2. *With the assumptions made above*, *whenever* $n^{-1}\tau_n \to \alpha \in (0,1]$ *in* P_{θ_0}*-probability and* $Q'(\theta_0) \neq 0$, then under P_{θ_0},

(2.16) $$\Lambda_{n,\tau_n} \to_w \Lambda \ \text{in}\ C(R)$$

where the process Λ *is defined in* (2.13).

Even though the limiting process Λ in Theorem 2 has sample paths in $C_0(R)$ -- the space of all real valued continuous functions on R vanishing at $\pm\infty$ -- the processes Λ_{n,τ_n} may not have paths in this space unless Θ is a bounded set. We define for each $\epsilon > 0$,

(2.17) $$\Lambda^{(\epsilon)}_{n,\tau_n}(u) = \Lambda_{n,\tau_n}(u), \text{ if } u \in (a_n,b_n) \cap (-\epsilon n^{\frac{1}{2}}, \epsilon n^{\frac{1}{2}})$$

and extend the definition of $\Lambda^{(\epsilon)}_{n,\tau_n}(u)$ to all of R in exactly the same way as in (2.8). Then $\Lambda^{(\epsilon)}_{n,\tau_n}$ has trajectories in $C_0(R)$. Note that the processes

$\Lambda^{(\varepsilon)}_{n,\tau_n}$ and Λ_{n,τ_n} are essentially the same if Θ is bounded. Equip $C_0(R)$ with the uniform metric topology.

Theorem 3. *With the assumptions made above, if* $n^{-1}\tau_n \to \alpha \in (0,1]$ *in* P_{θ_0}-*probability*, $Q'(\theta_0) \neq 0$ *and either*

(a) $P_{\theta_0}(\tau_n \geq [n\delta]) = 1$, for some $0 < \delta \leq \alpha$, all $n \geq 1$

or (b) $P_{\theta_0}(n^{-1}\tau_n < \alpha - \delta) \leq A^2\delta^n$, for some $A < \infty$ and $0 < \delta < \alpha$, all $n \geq 1$,

then for each $\varepsilon > 0$,

(2.18) $$\Lambda^{(\varepsilon)}_{n,\tau_n} \to_w \Lambda, \text{ in } C_0(R),$$

under the probability measure P_{θ_0}.

3. Proofs of Theorems. The conditional pdf of Z_i given $\mathcal{B}_{n,i-1}$ is

(3.1) $$q_\theta(z|\mathcal{B}_{n,i-1}) = (n-i+1)f_\theta(z)\{G_\theta(z)\}^{n-i}/\{G_\theta(Z_{i-1})\}^{n-i+1}$$

defined for $z > Z_{i-1}$. In view of (2.2) and (2.3) we then have for each i, $1 \leq i \leq n$

(3.2) $$q_\theta(Z_i/\mathcal{B}_{n,i-1}) = (n-i+1)h(Z_i)Q^{-1}(\theta)\exp(-(n-i+1)Q^{-1}(\theta)\int_{Z_{i-1}}^{Z_i} h(x)d\mu).$$

Define the random variables $Y_0, Y_1, \ldots, Y_n$, $n \geq 0$ as follows.

(3.3) $$Y_0 = 0,\ Y_i = (n-i+1)\int_{Z_{i-1}}^{Z_i} h\,d\mu,\ i = 1,\ldots,n.$$

Then it follows from (3.2) that the Y_i's are iidrv's with the simple exponential distribution and

(3.4) $$E_\theta(Y_i) = Q(\theta),\ \mathrm{Var}_\theta(Y_i) = Q^2(\theta),\ i = 1,\ldots,n.$$

Now from (2.5) and (3.1) one obtains for each k, $1 \leq k \leq n$

$$(3.5)\qquad p_\theta(\underset{\sim}{Z}^{(k)},n) = \prod_{i=1}^{k} q_\theta(Z_i|\mathcal{B}_{n,i-1}),$$

and therefore from (2.9) we get for any k, $1 \le k \le n$

$$(3.6)\qquad \xi_{n,k} = \sum_{i=1}^{k} \xi^*_{n,i},$$

where $\xi^*_{n,i} = \frac{\partial}{\partial\theta}(\log q_\theta(Z_i|\mathcal{B}_{n,i-1}))$, $i = 1,\ldots,n$.

Hence from (3.2) and (3.3), this gives us

$$\xi^*_{n,i} = -\frac{\partial}{\partial\theta}(\log Q(\theta)) + \frac{Q'(\theta)}{Q^2(\theta)} Y_i$$

$$(3.7)\qquad = \frac{Q'(\theta)}{Q^2(\theta)}(Y_i - Q(\theta)).$$

Employing (3.4) and the relation in (3.6) it also follows directly that

$$(3.8)\qquad J_{n,\tau_n}(\theta) = E_\theta\Big(\sum_{i=1}^{\tau_n} E_\theta(\xi^{*2}_{n,i}|\mathcal{B}_{n,i-1})\Big) = \left(\frac{Q'(\theta)}{Q(\theta)}\right)^2 E_\theta(\tau_n).$$

Therefore combining (2.12), (3.7) and (3.8) the process W_{n,τ_n} reduces to

$$(3.9)\qquad W_{n,\tau_n}(t;\theta) = \{E_\theta(\tau_n)\}^{-\frac{1}{2}} \sum_{i=1}^{[tE_\theta(\tau_n)]} (Y_i - Q(\theta))/Q(\theta).$$

Note that we are holding θ fixed in Θ. Since $n^{-1}\tau_n \to \alpha$ in P_θ-probability, by assumption, we have $n^{-1}E_\theta(\tau_n) \to \alpha$. Then since the $(Y_i - Q(\theta))/Q(\theta)$ are iid variables with zero mean and variance unity we obtain the desired result (2.15) by an application of Donsker's Theorem. (See Billingsley (1968)). Observe that

$$(3.10)\qquad \Lambda_{n,\tau_n}(u) = \prod_{i=1}^{\tau_n}\left(\frac{q_{\theta_n}(Z_i|\mathcal{B}_{n,i-1})}{q_{\theta_0}(Z_i|\mathcal{B}_{n,i-1})}\right)$$

where θ_n, $\theta_0 \in \Theta$ and θ_n is given by (2.6). Assume for the moment $\Theta = R$, i.e. $a = -\infty$, $b = +\infty$. Now for each i, $1 \le i \le n$ we have from (3.2)

$$\frac{q_{\theta_n}(Z_i|\mathcal{B}_{n,i-1})}{q_{\theta_0}(Z_i|\mathcal{B}_{n,i-1})} = \frac{Q(\theta_0)}{Q(\theta_n)} \exp\left(\frac{Q(\theta_n) - Q(\theta_0)}{Q(\theta_0)Q(\theta_n)}\right) Y_i \tag{3.11}$$

and therefore

$$\log\Lambda_{n,\tau_n}(u) = \frac{Q(\theta_n) - Q(\theta_0)}{Q(\theta_n)} \sum_{i=1}^{\tau_n}(Y_i - Q(\theta_0))/Q(\theta_0) + \tau_n\left(\frac{Q(\theta_n) - Q(\theta_0)}{Q(\theta_n)} - \log\frac{Q(\theta_n)}{Q(\theta_0)}\right). \tag{3.12}$$

But $Q(\theta_n) = Q(\theta_0) + un^{-\frac{1}{2}}Q'(\theta_n^*)$ where $|\theta_n^* - \theta_0| \le |u|n^{-\frac{1}{2}}$ and furthermore by the continuity of Q and Q'

$$Q(\theta_n) \to Q(\theta_0),\ Q'(\theta_n^*) \to Q'(\theta_0). \tag{3.13}$$

Hence the first term on the right hand side of (3.12) can be rewritten

$$un^{-\frac{1}{2}}\{Q'(\theta_n^*)/Q(\theta_n)\}\sum_{i=1}^{\tau_n}(Y_i - Q(\theta_0))/Q(\theta_0). \tag{3.14}$$

Again, since $n^{-1}\tau_n \to \alpha \in (0,1]$ in P_{θ_0}-probability and the $(Y_i - Q(\theta_0))/Q(\theta_0)$ are iidrv's with mean zero and unit variance it follows that, under the probability measure P_{θ_0}

$$\zeta_n = \alpha^{-1}n^{-\frac{1}{2}}\sum_{i=1}^{\tau_n}(Y_i - Q(\theta_0))/Q(\theta_0) \to_w \zeta, \tag{3.15}$$

where ζ is a standard normal variable. So in view of (2.14), (3.13) and (3.15) the entity in (3.14) converges weakly (under P_{θ_0}) to $uJ_\alpha^{\frac{1}{2}}(\theta_0)\zeta$.

To analyze the second term on the right hand side of (3.12) we proceed as follows. Let $x_n(u) = un^{-\frac{1}{2}}Q'(\theta_n^*)/Q(\theta_0)$. Now

$$\frac{Q(\theta_n) - Q(\theta_0)}{Q(\theta_n)} - \log \frac{Q(\theta_n)}{Q(\theta_0)}$$

$$= x_n(1 + x_n)^{-1} - \log(1 + x_n)$$

$$= x_n(1 + x_n)^{-1} - x_n + \tfrac{1}{2}x_n^2 - (\log(1 + x_n) - x_n + \tfrac{1}{2}x_n^2)$$

$$= -\tfrac{1}{2}x_n^2(1 + x_n)^{-1}(1 - x_n) - g_n, \text{ where}$$

$$g_n = g_n(u) = \log(1 + x_n(u)) - x_n(u) + \tfrac{1}{2}x_n^2(u). \tag{3.16}$$

From (3.13) and the definition of $x_n(u)$ we have at once $x_n(u) \to 0$ for each $u \in R$. Furthermore from the elementary inequalities

$$x^3/3(1 + x) \le \log(1 + x) - x + \tfrac{1}{2}x^2 \le x^3/3, \quad x > -1 \tag{3.17}$$

we also have that $ng_n(u) \to 0$ for each $u \in R$. Therefore since $n^{-1}\tau_n \to \alpha$ in P_{θ_0}-probability we get

$$\tau_n \left(\frac{Q(\theta_n) - Q(\theta_0)}{Q(\theta_n)} - \log \frac{Q(\theta_n)}{Q(\theta_0)}\right) = -\tfrac{1}{2}u^2 J_\alpha(\theta_0) + o_p(1) \tag{3.18}$$

and thus we have shown

$$\log\Lambda_{n,\tau_n}(u) \to_w \{uJ_\alpha^{\frac{1}{2}}(\theta_0)\zeta - \tfrac{1}{2}u^2 J_\alpha(\theta_0)\} = \log\Lambda(u) \tag{3.19}$$

for each fixed $u \in R$. Of course when $\Theta = (a,b)$, a,b finite (3.19) continues to hold for the process Λ_{n,τ_n} defined in (2.8). It also follows from (3.19) that the finite dimensional distributions of the process Λ_{n,τ_n} converge weakly to those of Λ. Therefore to complete the proof of Theorem 2 we need only verify that Λ_{n,τ_n} is tight. To demonstrate this it suffices to show that for any $L > 0$

$$E_{\theta_0}(\Lambda_{n,\tau_n}^{\frac{1}{2}}(u_1) - \Lambda_{n,\tau_n}^{\frac{1}{2}}(u_2))^2 \le K(u_1 - u_2)^2, \tag{3.20}$$

for all $u_1, u_2 \in [-L,L]$ and some constant $K > 0$ not depending on u_1, u_2

or n. However, since $E_{\theta_0}(\Lambda_{n,\tau_n}(u)) = 1$ for $u \in (a_n, b_n)$ and in view of (2.8) we need verify (3.20) for arbitrary $u_1, u_2 \in (a_n, b_n) \cap [-L, L]$. Now for u_1, u_2, with $u_2 < u_1$

$$(3.21)\qquad E_{\theta_0}(\Lambda^{\frac{1}{2}}_{n,\tau_n}(u_1) - \Lambda^{\frac{1}{2}}_{n,\tau_n}(u_2))^2 = \sum_{k=1}^{n} \int_{[\tau_n = k]} (p^{\frac{1}{2}}_{\theta_{n,1}}(\underset{\sim}{z}^{(k)}, n) - p^{\frac{1}{2}}_{\theta_{n,2}}(\underset{\sim}{z}^{(k)}, n))^2 d\mu_k$$

where μ_k is Lebesque measure in R^k and $\theta_{n,i} = \theta_0 + u_i n^{-\frac{1}{2}}$, $i = 1,2$. Further for each k, $1 \le k \le n$, by the Schwarz inequality

$$(3.22)\qquad \int_{[\tau_n = k]} (p^{\frac{1}{2}}_{\theta_{n,1}}(\underset{\sim}{z}^{(k)}, n) - p^{\frac{1}{2}}_{\theta_{n,2}}(\underset{\sim}{z}^{(k)}, n))^2 d\mu_k =$$

$$\int_{[\tau_n = k]} d\mu_k \left(\int_{\theta_{n,2}}^{\theta_{n,1}} \tfrac{1}{2} \frac{\partial p_\theta}{\partial \theta} p_\theta^{-\frac{1}{2}} d\mu(\theta) \right)^2$$

$$\le 1/4[\theta_{n,1} - \theta_{n,2}] \left(\int_{\theta_{n,2}}^{\theta_{n,1}} d\mu(\theta) \int_{[\tau_n = k]} p_\theta^{-1} \left(\frac{\partial p_\theta}{\partial \theta}\right)^2 d\mu_k \right).$$

Therefore, since $Q'(\theta)/Q(\theta)$ is bounded on Θ we have from (3.8) and (3.22) that the right hand side of (3.21) is dominated by $K^2(u_1 - u_2)^2/4$, where $K = \sup(Q'(\theta)/Q(\theta))$. This completes the proof of Theorem 2.

From our analysis of the process Λ_{n,τ_n} we also find that for each $\varepsilon > 0$ the finite dimensional distributions of the process $\Lambda^{(\varepsilon)}_{n,\tau_n}$ converge weakly under P_{θ_0} to the corresponding finite dimensional distributions of Λ. Thus once again we are left with verifying the tightness of $\Lambda^{(\varepsilon)}_{n,\tau_n}$ in $C_0(R)$. To this end we follow Ibragimov and Khas'minskii (1972,1975) and establish two preliminary lemmata.

Lemma 1. For each $\theta \in \Theta$, $n \ge 1$

$$(3.23)\qquad \sum_{k=1}^{n} \int_{[\tau_n = k]} \left(p^{\frac{1}{2}}_{\theta+h}(\underset{\sim}{z}^{(k)}, n) - p^{\frac{1}{2}}_{\theta}(\underset{\sim}{z}^{(k)}, n)\right)^2 d\mu_k = \frac{h^2}{4} J_{n,\tau_n}(\theta) + o(h^2)$$

as $h \to 0$.

Proof. Write $M(h)$ for the expression on the left hand side of (3.23). Following the usual argument as in (3.22) we have

$$M(h) \leq \frac{h}{4}\int_{\theta}^{\theta+h} J_{n,\tau_n}(\theta)\,d\mu(\theta).$$

and so from (3.8) we arrive at

$$\overline{\lim_{h\to 0}}\, h^{-2}M(h) \leq 1/4 J_{n,\tau_n}(\theta).$$

To obtain the reverse inequality we use Fatou's lemma.

$$\underline{\lim_{h\to 0}}\, h^{-2}M(h) \geq \sum_{k=1}^{n}\int_{[\tau_n=k]} 1/4 p_\theta^{-1}\left(\frac{\partial p_\theta}{\partial\theta}\right)^2 d\mu_k = 1/4 J_{n,\tau_n}(\theta),$$

and hence (3.23) is proven.

Lemma 2. Under the conditions placed on the sequence $\{\tau_n\}$ in Theorem 3, for any $K > 0$ there exists constants $c_1, c_2 > 0$ and an integer $n_0 \geq 1$ such that

$$P_{\theta_0}[\Lambda_{n,\tau_n}(u) \geq \exp(-c_1u^2)] \leq c_2\exp(-c_1u^2) \tag{3.24}$$

whenever $|u| \leq Kn^{\frac{1}{2}}$ and $n \geq n_0$.

Proof. In view of (2.8) we may confine attention to $u \in (a_n, b_n)$. Now $P_{\theta_0}[\Lambda_{n,\tau_n}(u) \geq \exp(-c_1u^2)] \leq \exp(\frac{1}{2}c_1u^2)\varphi_n$ where $\varphi_n = E_{\theta_0}(\Lambda_{n,\tau_n}^{\frac{1}{2}}(u))$. Since $E_{\theta_0}(\Lambda_{n,\tau_n}(u)) = 1$ we have from Lemma 1

$$\varphi_n = 1 - 1/8u^2n^{-1}J_{n,\tau_n}(\theta_0) + o(u^2/n). \tag{3.25}$$

Now $n^{-1}J_{n,\tau_n}(\theta_0) \to J_\alpha(\theta_0)$ as $n \to \infty$ and so for some integer $n_0 \geq 1$, $n^{-1}J_{n,\tau_n}(\theta_0) \geq \frac{1}{2}J_\alpha(\theta_0)$ whenever $n \geq n_0$. Also there exists $K^* > 0$, sufficiently small such that $|o(u^2/n)| \leq (u^2/32)J_\alpha(\theta_0)$, whenever $|u| \leq K^*n^{\frac{1}{2}}$. Hence for any u satisfying $|u| \leq K^*n^{\frac{1}{2}}$ with $n \geq n_0$ we obtain from (3.25)

$$\varphi_n \le (1 - \frac{u^2}{32} J_\alpha(\theta_0)) \le \exp(-\frac{J_\alpha(\theta_0)}{32} u^2)$$

and so (3.24) follows by choosing $c_2 = 1$ and $c_1 > 0$ appropriately.

Suppose $K^* n^{\frac{1}{2}} \le |u| \le Kn^{\frac{1}{2}}$, with $K > 0$ arbitrary. Note that $\{\Lambda^{\frac{1}{2}}_{n,k}, \mathcal{B}_{n,k}\}^n_{k=1}$ is a nonnegative supermartingale under P_{θ_0} so that under condition (a) of Theorem 3, viz $\tau_n \ge [n\delta]$ a.s. (P_{θ_0}) for some $0 < \delta \le \alpha$, we get

$$(3.26) \qquad \varphi_n = E_{\theta_0}(\Lambda^{\frac{1}{2}}_{n,\tau_n} I(\tau_n \ge [n\delta])) \le E_{\theta_0}(\Lambda^{\frac{1}{2}}_{n,[n\delta]}) = (\varphi(un^{-\frac{1}{2}}))^{[n\delta]}$$

where $\varphi(x) = 2\sqrt{Q(\theta_0)Q(\theta_0 + x)}/(Q(\theta_0) + Q(\theta_0 + x))$. Since φ is continuous in x and $\varphi(x) < 1$ whenever $x \neq 0$ we have $\varphi_0 = \sup_{x \in [K^*,K]} \varphi(x) < 1$ and therefore

$$\varphi_n \le \exp(-[n\delta]|\log\varphi_0|) \le \exp(-\frac{u^2\delta}{2K^2}|\log\varphi_0|),$$

for all n sufficiently large and again (3.24) follows. Finally if condition (b) of Theorem 3 holds, we obtain

$$\varphi_n \le E_{\theta_0}(\Lambda^{\frac{1}{2}}_{n,\tau_n} I(\tau_n \ge [n(\alpha - \delta)])) + E_{\theta_0}(\Lambda^{\frac{1}{2}}_{n,\tau_n} I(\tau_n/n - \alpha < -\delta))$$

$$(3.27) \qquad \le (\varphi(un^{-\frac{1}{2}}))^{[n(\alpha-\delta)]} + A\delta^{n/2}$$

$$\le \exp(-\frac{u^2(\alpha-\delta)}{2K^2}|\log\varphi_0|) + A\exp(-\frac{u^2}{2K^2}|\log\delta|),$$

for large enough n and this leads us to (3.24). This completes the proof of the lemma.

The remainder of the proof follows Ibragimov and Khasminskii (1972). In view of (3.20) and Lemma 2, for any $\varepsilon > 0$ and $0 < \ell < \varepsilon n^{\frac{1}{2}}$

$$(3.28) \qquad P_{\theta_0}[\sup_{\substack{|u_1-u_2|<h \\ |u_i|<\ell}} |\Lambda^{\frac{1}{2}}_{n,\tau_n}(u_1) - \Lambda^{\frac{1}{2}}_{n,\tau_n}(u_2)| > h^{1/4}] \le ch^{\frac{1}{2}}$$

for some constant $c > 0$, and for arbitrary $\eta > 0$, there exists $a > 0$,

such that

$$\text{(3.29)} \qquad \lim_{n\to\infty} P_{\theta_0}\left[\sup_{a<|u|<\epsilon n^{\frac{1}{2}}} \Lambda^{\frac{1}{2}}_{n,\tau_n}(u) > \eta\right] = 0$$

Then the tightness of $\Lambda^{(\epsilon)}_{n,\tau_n}$ in $C_0(R)$ follows from (3.28) and (3.29).

4. Some Remarks and Examples: (a) The one-parameter processes Λ_{n,τ_n} and $\Lambda^{(\epsilon)}_{n,\tau_n}$ of Theorem 2 and 3 are those usually encountered in practice. For the sake of completeness however, we mention here briefly the two-parameter process $\{\tilde{\Lambda}_{n,k_n(t;\theta_0)}(u);\ t \in [0,1],\ u \in R\}$, defined in the usual way for $u \in (a_n, b_n)$ and set constant (> 0) otherwise, keeping the sample paths continuous in u. Note that throughout θ_0 is held fixed in Θ. With these definitions it is not too difficult to see that the finite dimensional distributions of the process $\tilde{\Lambda}_{n,\tau_n}$ converge weakly under P_{θ_0} to those of the process $\tilde{\Lambda} = \{\tilde{\Lambda}(u,t) : t \in [0,1],\ u \in R\}$ where

$$\tilde{\Lambda}(u,t) = \exp\{uJ_\alpha^{\frac{1}{2}}(\theta_0)W(t) - \tfrac{1}{2}u^2 tJ_\alpha(\theta_0)\}.$$

In order to obtain the weak convergence of the entire process $\tilde{\Lambda}_{n,\tau_n}$ we need to proceed further. We shall provide an outline here. Consider the space $D = D([0,1] \times R)$ of all real valued functions on $[0,1] \times R$ which are continuous from above with limits from below in the sense explained in Neuhaus (1971) or Bickel and Wichura (1971). For each $j \geq 1$ set $D_j = D([0,1] \times [-j,j])$ and let d_j be a metric on D_j generating the Skorohod topology there. To define a d_j consider the class Λ_0 (respectively Λ_j) of all strictly increasing continuous mappings on $[0,1]$ onto $[0,1]$ (respectively on $[-j,j]$ onto $[-j,j]$). Let $\lambda = (\lambda_0, \lambda_j) \in \Lambda_0 \times \Lambda_j$. For any $(t,u) \in [0,1] \times [-j,j]$ write $|(t,u)| = \max\{|t|,|u|\}$ and $\lambda(t,u) = (\lambda_0(t), \lambda_j(u))$. Then we may define d_j by

$d_j(x,y) = \inf\{\varepsilon > 0 :$ for some $\lambda = (\lambda_0, \lambda_j) \in \Lambda \times \Lambda_j$ with

$\|\lambda_0\| < \varepsilon$, $\|\lambda_j\| < \varepsilon$, $\sup_{(t,u)\in[0,1]\times[-j,j]} |x(t,u) - y(\lambda(t,u))| \le \varepsilon\}$

where $\|\lambda_0\| = \sup_{t\neq s} \left|\log \frac{\lambda_0 u - \lambda_0 \nu}{u - \nu}\right|$ and $\|\lambda_j\|$ is similarly defined. Then the metric d in D given by

$$d(x,y) = \sum_{j=1}^{\infty} 2^{-j} \frac{d_j(x,y)}{1 + d_j(x,y)}$$

converts D into a complete separable metric space.

The sample paths of $\tilde{\Lambda}_{n,\tau_n}$ lie in D and those of Λ lie in the subset $C([0,1] \times R)$ of D, consisting of all continuous functions on $[0,1] \times R$. To verify tightness of the process Λ_{n,τ_n} it suffices to show, for each $j \ge 1$, and arbitrary $\varepsilon > 0$

$$\lim_{\delta_1,\delta_2\to 0} \limsup_{n\to\infty} P_{\theta_0}\{\sup[|\log\tilde{\Lambda}_{n,k_n(t_1;\theta_0)}(u_1) - \log\tilde{\Lambda}_{n,k_n(t_2;\theta_0)}(u_2)|$$

$$: |u_1 - u_2| < \delta_1,\ |t_1 - t_2| < \delta_2,\ |u_i| \le j,\ t \in [0,1]] > \varepsilon\} = 0$$

To show this we follow (3.12) and write

$$\log\tilde{\Lambda}_{n,k_n(t;\theta_0)}(u) = u(n^{-\frac{1}{2}}E_{\theta_0}^{\frac{1}{2}}(\tau_n))\left(\frac{Q'(\theta_n^*)}{Q(\theta_0)}\right)W_{n,\tau_n}(t;\theta_0)$$

$$- \tfrac{1}{2}u^2(n^{-1}k_n(t;\,\theta_0))\left(\frac{Q'(\theta_n^*)}{Q(\theta_0)}\right)^2(1 + x_n)^{-1}(1 - x_n)$$

$$- (n^{-1}k_n(t;\,\theta_0))ng_n,$$

where θ_n^*, x_n and g_n are defined in (3.13) - (3.16). The remainder of the analysis follows the usual conventional manipulations. Note that the tightness of W_{n,τ_n} in $D[0,1]$ which follows from Theorem 1 will be used.

(b) The fundamental assumptions on the sequence $\{\tau_n\}$ are that they be adapted to the σ-fields $\{\mathcal{B}_{n,k};\ 1 \le k \le n\}$ and satisfy $n^{-1}\tau_n \to \alpha \in (0,1]$

in probability. Theorem 3 imposes a condition on the rate of this convergence. For example in the simplest situation where sampling is terminated at time $t > 0$ we may take $\tau_n = nF_n(t)$, where $F_n(t)$ is the empirical d.f. of $X_{n,1},\ldots,X_{n,n}$. In this case the inequality $P_{\theta_0}[|\frac{\tau_n}{n} - \alpha| \geq \delta] \leq A^2\delta^n$, holds for some constants $A > 0$ and $0 < \delta < 1$ with $\alpha = F_\theta(t)$. Gardiner and Sen (1978) have considered a wider class of stopping variables τ_n that are expressible in terms of certain linear combinations of functions of the observables $X_{n,1},\ldots,X_{n,n}$ which is appropriate to this context.

(c) The restriction imposed in this paper to classes of distributions satisfying (2.3) enables us to work in terms of independent variables even though the observables $X_{n,1},\ldots,X_{n,n}$ are dependent. However, if this condition does not hold results paralleling those given here can be obtained though the analysis is essentially different and necessarily more involved as one has lost the enormous technical facility of working with independent random variables. In particular the transformation (3.3) cannot be made and $J_{n,\tau_n}(\theta)$ of (3.8) takes on a far more complicated form even though as $n \to \infty$ $n^{-1}J_{n,\tau_n}(\theta)$ converges to a limit. Finally, we remark that randomly stopped likelihood ratio processes can be analysed for general dependent triangular arrays $\{X_{n,k} : 1 \leq k \leq k_n, n \geq 1\}$ with a different choice of local coordinates θ_n.

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The weak convergence of certain (randomly stopped) likelihood ratio processes based on the ordered observations corresponding to a random sample is considered in the situation where the hazard rate function of the underlying distribution is separable in its variables. It is shown that under mild conditions on the stopping variables the log-likelihood function is locally asymptotically normal. Some remarks pertaining to the general ease and applications of the theorems proved are also discussed.

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